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Restrictions of Fourier Transforms and Extension of Fourier Sequences

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In this paper we discuss relations between Fourier–Stieltjes transforms of (bounded Borel) measures on $(-\infty, \infty)$ and sequences of Fourier–Stieltjes coefficients of measures on $[0, 2\pi)$. We show (Theorem B) that if $\hat{\mu}$ is the Fourier–Stieltjes transform of a measure μ on $(-\infty, \infty)$, then $\{\hat{\mu}(n)\}_{n=-\infty}^{\infty}$ is the sequence of Fourier–Stieltjes coefficients of some measure on $[0, 2\pi)$. In the other direction we show (Theorem F) that if $\{a_n\}_{n=-\infty}^{\infty}$ is the sequence of Fourier–Stieltjes coefficients of some measure on $[0, 2\pi)$, then the function whose graph consists of the line segments successively joining the points (n, a_n) must be the Fourier–Stieltjes transform of a measure on $(-\infty, \infty)$. We also prove a similar theorem for distributions.

I. Let $M(-\infty, \infty)$ and $M[0, 2\pi)$ denote the spaces of bounded Borel measures on $(-\infty, \infty)$ and $[0, 2\pi)$, respectively. If $\mu \in M(-\infty, \infty)$, its Fourier–Stieltjes transform $\hat{\mu}$ is defined as

$$\hat{\mu}(x) = \int_{-\infty}^{\infty} e^{-ixt}\, d\mu(t) \qquad (-\infty < x < \infty).$$

If $\nu \in M[0, 2\pi)$, the sequence $\{a_n\}_{n=-\infty}^{\infty}$ of Fourier–Stieltjes coefficients of ν is given by

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-int}\, d\nu(t) \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

The abbreviation FT stands for Fourier–Stieltjes transform of a measure on $(-\infty, \infty)$ and FS stands for the sequence of Fourier–Stieltjes coefficients of a measure on $[0, 2\pi)$.

We shall make use of the functions δ and Δ, given by

$$\delta(t) = \frac{1}{\pi} \cdot \frac{1 - \cos t}{t^2} \qquad (-\infty < t < \infty) \tag{1}$$

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and

$$\Delta(x) = 1 - |x| \qquad (|x| \leqslant 1)$$
$$\Delta(x) = 0 \qquad (|x| > 1).$$

It is well-known [2], p. 21 that $\delta \in L^1(-\infty, \infty)$ and $\Delta = \hat{\delta}$.

II. The first theorem shows that if h is a FT, then $\{h(n)\}_{n=-\infty}^{\infty}$ is a FS. (For absolutely continuous measures this is essentially given in [5], p. 68). We need first

LEMMA A. *Let $\mu \in M(-\infty, \infty)$ and for $E \subset [0, 2\pi)$,* let

$$\nu(E) = 2\pi \sum_{k=-\infty}^{\infty} \mu(E + 2k\pi). \tag{2}$$

Then $\nu \in M[0, 2\pi)$.

Proof. To show ν is bounded, we have

$$\frac{1}{2\pi}\int_0^{2\pi} d\,|\,\nu\,|\,(t) \leqslant \int_0^{2\pi} d\left(\sum_{k=-\infty}^{\infty} |\,\mu\,|\,(t + 2k\pi)\right) = \sum_{k=-\infty}^{\infty}\int_0^{2\pi} d\,|\,\mu\,|\,(t + 2k\pi)$$
$$= \sum_{k=-\infty}^{\infty}\int_{2k\pi}^{(2k+2)\pi} d\,|\,\mu\,|\,(t) = \int_{-\infty}^{\infty} d\,|\,\mu\,|\,(t) < \infty.$$

The interchange of sum and integral is justified by absolute convergence.

THEOREM B. *If $\mu \in M(-\infty, \infty)$ and ν is defined by* (2), *then $\{\hat{\mu}(n)\}_{n=-\infty}^{\infty}$ is the* FS *for ν.*

Proof. We have, for $n = 0, \pm 1, \pm 2, \ldots$,

$$\hat{\mu}(n) = \int_{-\infty}^{\infty} e^{-int}\, d\mu(t) = \sum_{k=-\infty}^{\infty}\int_{2k\pi}^{(2k+2)\pi} e^{-int}\, d\mu(t)$$
$$= \sum_{k=-\infty}^{\infty}\int_0^{2\pi} e^{-int}\, d\mu(t + 2k\pi) = \frac{1}{2\pi}\int_0^{2\pi} e^{-int}\, d\nu(t) = \hat{\nu}(n).$$

The required absolute convergence follows as in the proof of the lemma.

Remark C. It is clear that if μ is absolutely continuous, then so is ν. Thus, if $f \in L^1(-\infty, \infty)$ and

$$\varphi(t) = 2\pi \sum_{k=-\infty}^{\infty} f(t + 2k\pi)$$

then the series converges for almost all $t \in [0, 2\pi)$, and $\varphi \in L^1[0, 2\pi)$. Moreover, $\{\hat{f}(n)\}_{n=-\infty}^{\infty}$ will be the Fourier coefficients for φ. Thus, if f, $g \in L^1(-\infty, \infty)$ and $\hat{f}(n) = \hat{g}(n)$ for all n, then

$$\sum_{k=-\infty}^{\infty} f(t + 2k\pi) = \sum_{k=-\infty}^{\infty} g(t + 2k\pi) \quad \text{a.e.} \quad (0 \leqslant t \leqslant 2\pi)$$

(since the Fourier coefficients for both sums will be equal). Here is an application. Let

$$\psi(t) = 1/2\pi \qquad (0 \leqslant t < 2\pi),$$
$$\psi(t) = 0 \qquad \text{for all other } t.$$

Then $\psi \in L^1(-\infty, \infty)$ and $\hat{\psi}(0) = 1$, $\hat{\psi}(n) = 0$ for $n \neq 0$. Thus,

$$\hat{\psi}(n) = \Delta(n) = \hat{\delta}(n)$$

for all n, and so

$$\sum_{k=-\infty}^{\infty} \delta(t + 2k\pi) = \sum_{k=-\infty}^{\infty} \psi(t + 2k\pi) \quad \text{a.e.}$$

But the sum on the right is easily seen to be equal to $1/2\pi$ for all t. Thus, $\sum_{k=-\infty}^{\infty} \delta(t + 2k\pi) = 1/2\pi$ a.e. That is,

$$\frac{1}{\pi} \sum_{k=-\infty}^{\infty} \frac{1 - \cos t}{(t + 2k\pi)^2} = \frac{1}{2\pi}.$$

Changing t to $2t$, we deduce the familiar identity

$$\sum_{k=-\infty}^{\infty} \frac{1}{(t + k\pi)^2} = \frac{1}{\sin^2 t}.$$

III. If g is a function on $(-\infty, \infty)$, we have seen that a necessary condition that g be a FT is that $\{g(n)\}_{n=-\infty}^{\infty}$ be a FS. This condition is clearly not sufficient, since g can behave badly on the noninteger reals. However, there is one type of functions g on $(-\infty, \infty)$ for which the condition that $\{g(n)\}_{n=-\infty}^{\infty}$ be a FS is sufficient in order that g be a FT—namely, functions which are linear on each interval $[n, n+1]$.

LEMMA D. *Let $\{a_n\}_{n=-\infty}^{\infty}$ be any sequence of complex numbers, and let*

$$g(x) = \sum_{k=-\infty}^{\infty} a_k \Delta(x - k) \qquad (-\infty < x < \infty), \tag{3}$$

Then $g(n) = a_n$ $(n = 0, \pm 1, \pm 2, \ldots)$ and g is linear on each interval $[n, n+1]$.

Proof. Fix n. If $n \leqslant x \leqslant n+1$, then, since $\Delta(y)$ vanishes outside $-1 < y < 1$, $\Delta(x-k) = 0$ for all $k \leqslant n-1$ and all $k \geqslant n+2$. That is, if $n \leqslant x \leqslant n+1$, then

$$\begin{aligned} g(x) &= a_n \Delta(x-n) + a_{n+1}\Delta(x-n-1) \\ &= a_n[1 - |x-n|] + a_{n+1}[1 - |x-n-1|] \\ &= a_n[1-(x-n)] + a_{n+1}[1-(n+1-x)]. \end{aligned}$$

Thus, $g(n) = a_n$ and g is linear on $[n, n+1]$.

Restated, the lemma says that the function whose graph consists of the line segments joining (n, a_n) to $(n+1, a_{n+1})$, for $n = 0, \pm 1, \pm 2, \ldots$, is given by (3).

LEMMA E. *Let $\nu \in M[0, 2\pi)$ and let $\nu^{\sim}$ be the periodic extension of ν to $(-\infty, \infty)$. That is, for $E \subset [2k\pi, (2k+2)\pi)$,*

$$\nu^{\sim}(E) = \nu(E - 2k\pi).$$

Let $\mu = \delta\nu^{\sim}$. (That is, $\mu(E) = \int_E \delta(t)\, d\nu^{\sim}(t)$.) Then $\mu \in M(-\infty, \infty)$.

Proof.

$$\int_{-\infty}^{\infty} d\,|\mu|\,(t) = \int_{-\infty}^{\infty} \delta(t)\, d\,|\nu^{\sim}|\,(t) = \sum_{k=-\infty}^{\infty} \int_{2k\pi}^{(2k+2)\pi} \delta(t)\, d\,|\nu^{\sim}|\,(t)$$

$$= \sum_{k=-\infty}^{\infty} \int_0^{2\pi} \delta(t+2k\pi)\, d\,|\nu^{\sim}|\,(t+2k\pi) = \sum_{k=-\infty}^{\infty} \int_0^{2\pi} \delta(t+2k\pi)\, d\,|\nu|\,(t)$$

$$\leqslant A \sum_{k=-\infty}^{\infty} M_k, \qquad \text{where} \quad M_k = \max_{0 \leqslant t \leqslant 2\pi} \delta(t+2k\pi).$$

Clearly, $M_k = O(k^{-2})$ as $k \to \infty$ and so, $\sum_{k=-\infty}^{\infty} M_k < \infty$, which proves the lemma.

THEOREM F. *Let $\{a_n\}_{n=-\infty}^{\infty}$ be the* FS *for $\nu \in M[0, 2\pi)$, and let g be the function whose graph consists of the line segments joining successively the points (n, a_n), for $n = 0, \pm 1, \pm 2, \ldots$. Then g is a* FT*—indeed, g is the* FT *of the measure $\mu = \delta\nu^{\sim}$.*

Proof. From Lemma D we have, for any x,

$$g(x) = \sum_{n=-\infty}^{\infty} a_n \Delta(x-n) = \sum_{n=-\infty}^{\infty} a_n \int_{-\infty}^{\infty} e^{-i(x-n)t}\, \delta(t)\, dt$$

$$= \sum_{n=-\infty}^{\infty} a_n \int_{-\infty}^{\infty} e^{-ixt}\, \delta(t)\, e^{int}\, dt. \tag{4}$$

We would like to interchange sum and integral. However, we do not know even that $\sum_{n=-\infty}^{\infty} a_n e^{int}$, which is the Fourier–Stieltjes series for ν, converges. Hence, we use $(C, 1)$ summability. Since the series on the right of (4) converges to $g(x)$, it is $(C, 1)$ summable to $g(x)$. That is,

$$g(x) = \lim_{N\to\infty} \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) a_n \int_{-\infty}^{\infty} e^{-ixt}\, \delta(t)\, e^{int}\, dt$$
$$= \lim_{N\to\infty} \int_{-\infty}^{\infty} e^{-ixt}\, \delta(t) \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) a_n e^{int}\, dt,$$

and so

$$g(x) = \lim_{N\to\infty} \int_{-\infty}^{\infty} e^{-ixt}\, \delta(t)\, \sigma_N(t)\, dt \qquad (-\infty < x < \infty), \tag{5}$$

where the σ_N are the $(C, 1)$ means of $\sum_{n=-\infty}^{\infty} a_n e^{int}$. Now, σ_N converges weak* on the circle to ν [3], p. 20. That is,

$$\lim_{N\to\infty} \int_0^{2\pi} F(t)\, \sigma_N(t)\, dt = \int_0^{2\pi} F(t)\, d\nu(t),$$

for F continuous on $[0, 2\pi]$, with $F(0) = F(2\pi)$. Hence,

$$\lim_{N\to\infty} \int_{2k\pi}^{(2k+2)\pi} G(t)\, \sigma_N(t)\, dt = \int_{2k\pi}^{(2k+2)\pi} G(t)\, d\nu^{\sim}(t),$$

for any G continuous on $[2k\pi, (2k+2)\pi]$, with $G(2k\pi) = G[(2k+2)\pi]$, and so

$$\lim_{N\to\infty} \int_{2k\pi}^{(2k+2)\pi} e^{-ixt}\, \delta(t)\, \sigma_N(t)\, dt = \int_{2k\pi}^{(2k+2)\pi} e^{-ixt}\, \delta(t)\, d\nu^{\sim}(t), \tag{6}$$

for $k = 0, \pm 1, \pm 2, \ldots$.(Note $\delta(2k\pi) = \delta[(2k+2)\pi] = 0$.) Continuing, we have

$$\int_{-\infty}^{\infty} e^{-ixt}\, \delta(t)\, \sigma_N(t)\, dt = \sum_{k=-\infty}^{\infty} \int_{2k\pi}^{(2k+2)\pi} e^{-ixt}\, \delta(t)\, \sigma_N(t)\, dt. \tag{7}$$

Now $\int_{2k}^{(2k+2)\pi} |\sigma_N(t)|\, dt \leqslant M$, for all $N = 1, 2, \ldots$, [3], p. 23. For each $k = 0, \pm 1, \pm 2, \ldots$, we thus have $\int_{2k\pi}^{(2k+2)\pi} |e^{-ixt}\, \delta(t)\, \sigma_N(t)|\, dt \leqslant MM_k$, where M_k is as in the proof of Lemma E. Hence, the series on the right of (7) is term by term dominated by $\sum_{k=-\infty}^{\infty} MM_k < \infty$, which is independent of N. We may, thus, let $N \to \infty$ under the summation sign in (7). This and (6) give

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} e^{-ixt}\, \delta(t)\, \sigma_N(t)\, dt = \sum_{k=-\infty}^{\infty} \int_{2k\pi}^{(2k+2)\pi} e^{-ixt}\, \delta(t)\, d\nu^{\sim}(t)$$
$$= \int_{-\infty}^{\infty} e^{-ixt}\, \delta(t)\, d\nu^{\sim}(t).$$

Hence, by (5), g is the FT of $\mu = \delta\nu^{\sim}$, which is what we wished to show.

From the definition of μ, it is clear that if ν is absolutely continuous, then so is μ. We, thus, have

COROLLARY G. *Let $\varphi \in L^1[0, 2\pi)$ have Fourier coefficients $\{a_n\}_{n=-\infty}^{\infty}$ and let g be the function whose graph consists of the line segments joining successively the points (n, a_n), for $n = 0, \pm 1, \pm 2, \ldots$. Then g is the Fourier transform of a function in $L^1(-\infty, \infty)$—indeed, $g = (\delta\varphi^{\sim})^{\wedge}$, where $\varphi^{\sim}$ is the periodic extension to $(-\infty, \infty)$ of φ.*

IV. For those familiar with distributions we can give a generalizaion of Theorem F. For distributions on the circle we refer the reader to [1], Chap. 12 and for distributions on the line, to [4], Chap. 4.

THEOREM H. *Let T be a distribution on $[0, 2\pi)$ having Fourier coefficients $\{a_n\}_{n=-\infty}^{\infty}$ and let g be the function whose graph consists of the line segments successively joining the points (n, a_n) for $n = 0, \pm 1, \pm 2, \ldots$. Then g (considered as a distribution) is the Fourier transform of a tempered distribution—indeed $g = (\delta T^{\sim})^{\wedge}$, where $T^{\sim}$ is the periodic extension of T to $(-\infty, \infty)$.*

Proof. It is known that every distribution on the circle has finite order. Hence,

$$a_n = O(|n|^s) \qquad (|n| \to \infty), \tag{8}$$

for some positive integer s. Now, $T^{\sim}$ can be defined as

$$T^{\sim} = \sum_{n=-\infty}^{\infty} a_n e^{int} \qquad (-\infty < t < \infty)$$

and, because of (8), $T^{\sim}$ will be a tempered distribution on $(-\infty, \infty)$. If e_n is the measure on $(-\infty, \infty)$, with mass 1 at n, and if

$$\varphi = \sum_{n=-\infty}^{\infty} a_n e_n ,$$

then φ is a tempered distribution and is the Fourier transform of $T^{\sim}$. By Lemma E,

$$g(x) = \sum_{n=-\infty}^{\infty} a_n \Delta(x-n)$$

and so, since $e_n * \Delta = \Delta(x - n)$,

$$g = \left(\sum_{n=-\infty}^{\infty} a_n e_n\right) * \Delta = \varphi * \Delta,$$

where g is, now, a distribution.

But ([4], p. 424), if T_1 is a tempered distribution and T_2 is a tempered distribution whose Fourier transform is a function with compact support, then $T_1\hat{} * T_2\hat{} = (T_1T_2)\hat{}$. Hence, since Δ has compact support, we have $g = \varphi * \Delta = T^{\sim\wedge} * \delta^{\wedge} = (T^{\sim}\delta)^{\wedge}$ and the proof is complete.

If T is a measure, then Theorem H together with Lemma E gives a shorter proof of Theorem F. However, the results concerning distributions that we have used are much deeper and more difficult to establish than the results used in the proof of Theorem F, as given.

References

1. R. E. Edwards, "Fourier Series, A Modern Introduction," Vol. II, Holt Rinehart Winston, New York, 1967.
2. R. R. Goldberg, "Fourier Transforms," Cambridge University Press, Cambridge, 1961.
3. K. Hoffman, "Banach Spaces of Analytic Functions," Prentice Hall, Englewood Cliffs, N. J., 1962.
4. J. Horvath, "Topological Vector Spaces and Distributions," Vol. I, Addison Wesley, Reading, Mass., 1966.
5. A. Zygmund, "Trigonometric Series," Vol. I, Cambridge University Press, Cambridge, 1959.